

A NOTE ON THE EXTENDED dTODA HIERARCHY

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Abstract

We give a derivation of dispersionless Hirota equations for the extended dispersionless Toda hierarchy. We show that the dispersionless Hirota equations are nothing but a direct consequence of the genus-zero topological recursion relation for the topological CP^1 model. Using the dispersionless Hirota equations we compute the two point functions and express the result in terms of Catalan number.

Keywords: extended dToda hierarchy, dispersionless Hirota equation, Catalan number, topological field theory.

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1 Introduction

Recently, Kodama and Pierce[12] gave a combinatorial description of the one-dimensional dispersionless Toda(dToda) hierarchy to solve the two-vertex problem on a sphere. The main strategy is to characterize the free energy $F(t_0, t)$ ($t = (t_1, t_2, \dots)$) of the dToda hierarchy by the corresponding dispersionless Hirota equations. Then the second derivatives of the free energy $\partial_{t_n} \partial_{t_m} F \equiv F_{n,m}$ satisfy a set of algebraic relations. Surprisingly they found a closed form for the rational numbers $F_{n,m}$ under the conditions $F_{01} = F_{00} = 0$ for general n and m . In particular, the formulas of $F_{n,m}$ can be expressed in terms of the Catalan number which is commonly used in the context of enumerative combinatorics (see e.g. [18]). Their result for $F_{n,m}$ provides a combinatorial meaning of a counting problem of connected ribbon graphs with two vertices of degree n and m on a sphere and is a generalization of the previous works where the problem has been solved only in the case of the same degree (that is F_{nn}) [10, 15].

In this work, motivated by the aforementioned result, we like to generalize the computation of the two point functions $F_{n,m}$ to the extended dToda hierarchy[8, 7, 5, 6] which is an extension of the one-dimensional dToda hierarchy by adding logarithmic type conserved densities. Since extended dToda hierarchy is the dispersionless limit of the extended Toda hierarchy [23, 2] which has been used to govern the Gromov-Witten(GW) invariants(see e.g. [9] and references therein) for the CP^1 manifold. Thus the extended dToda hierarchy becomes the master equation of the genus zero GW invariants whose generating function is characterized by the free energy of the extended dToda hierarchy. Based on the twistor theoretical method [20, 11] the extended dToda hierarchy can be constructed by adding logarithmic-flow to the one-dimensional dToda hierarchy. The corresponding Orlov-Schulman operator is conjugated with the Lax operator under the Poisson bracket which imposes an extra condition (the so-called string equation) on the free energy of the extended dToda hierarchy. We will show that the full hierarchy flows can be expressed in terms of second derivatives of its associated free energy F and thus can be viewed as the corresponding dispersionless Hirota(dHirota) equations. We then investigate the two point functions of the extended dToda hierarchy based on the associated dHirota equations and express the result in terms of the Catalan number. To make a connection with the topological field theory, we rewrite the dHirota equations in CP^1 time parameters and show that they are indeed a direct consequence of the genu-zero topological recursion relation[21] of the topological CP^1 model.

This paper is organized as follows. In section 2, we recall the Lax formalism of the extended dToda hierarchy. In section 3, we derive the dHirota equation of the extended dToda hierarchy which can be expressed as a set of equations in terms of second derivatives of the free energy. The initial values of two-point functions of the extended dToda hierarchy are computed in Section 4. In section 5, we reinterpret the dHirota equations from topological field theory point of view. Section 6 is devoted to the concluding remarks.

2 The extended dispersionless Toda hierarchy

The one-dimensional dToda hierarchy [20, 12] is defined by the Lax equation

$$\frac{\partial L}{\partial t_n} = \{B_n, L\}, \quad B_n = (L^n)_{\geq 0}.$$

where L is a two-variable Lax operator of the form

$$L = p + u_1 + u_2 p^{-1} \quad (1)$$

with u_1 and u_2 are functions of the time variables $t = (t_1, t_2, \dots)$ along with a spatial variable t_0 . Here $(A)_{\geq 0}$ denotes the polynomial part of A , $(A)_{\leq -1} = A - (A)_{\geq 0}$, and the Poisson bracket $\{, \}$ is defined by

$$\{A(p, t_0), B(p, t_0)\} = p \frac{\partial A(p, t_0)}{\partial p} \frac{\partial B(p, t_0)}{\partial t_0} - p \frac{\partial A(p, t_0)}{\partial t_0} \frac{\partial B(p, t_0)}{\partial p}.$$

In particular, the fundamental variable u_1 and u_2 can be expressed in terms of second derivatives of F as

$$u_1 = F_{01}, \quad u_2 = F_{11} = e^{F_{00}}$$

where the second equation is just the one-dimensional reduction of the dToda field equation. Following the twistor theoretical construction [20, 11], the extended dToda hierarchy can be constructed from the one-dimensional dToda hierarchy by adding the \hat{t}_n -flows as

$$\frac{\partial L}{\partial \hat{t}_n} = \{\hat{B}_n, L\}, \quad \hat{B}_n = (L^n (\log L - d_n))_{\geq 0} \quad (2)$$

where $d_n = \sum_{j=1}^n 1/j$ with $d_0 \equiv 0$ and $\log L$ is defined by the prescription

$$\log L = \frac{1}{2} \log u_2 + \frac{1}{2} \log(1 + u_1 p^{-1} + u_2 p^{-2}) + \frac{1}{2} \log \left(1 + \frac{u_1}{u_2} p + \frac{1}{u_2} p^2 \right) \quad (3)$$

with the proviso that we shall Taylor expand the second term in p^{-1} , whereas in p for the last term. Moreover, the associated Orlov-Schulman is given by

$$N(t_0, t, \hat{t}) = \sum_{n=1}^{\infty} n t_n L^n + t_0 + \sum_{n=1}^{\infty} n \hat{t}_n L^n (\log L - d_{n-1}) + \sum_{n=1}^{\infty} F_{n0} L^{-n},$$

which satisfies

$$\partial_{t_n} N = \{B_n, N\}, \quad \partial_{\hat{t}_n} N = \{\hat{B}_n, N\}, \quad \{L, N\} = L.$$

The symplectic two-form of the extended dToda hierarchy can be written as

$$\omega \equiv \frac{dp}{p} \wedge dt_0 + \sum_{n=1}^{\infty} dB_n \wedge dt_n + \sum_{n=1}^{\infty} d\hat{B}_n \wedge d\hat{t}_n = \frac{dL \wedge dN}{L}$$

which implies the existence of a S function such that

$$dS(t_0, t, \hat{t}) = N d \log L + \log p dt_0 + \sum_{n=1}^{\infty} B_n dt_n + \sum_{n=1}^{\infty} \hat{B}_n d\hat{t}_n$$

or, equivalently,

$$N = \frac{\partial S}{\partial \log L}, \quad \log p = \frac{\partial S}{\partial t_0}, \quad B_n = \frac{\partial S}{\partial t_n}, \quad \hat{B}_n = \frac{\partial S}{\partial \hat{t}_n}.$$

It is not hard to show that the S function has the form

$$S = \sum_{n=1} t_n L^n + t_0 \log L + \sum_{n=1} \hat{t}_n L^n (\log L - d_n) - \sum_{n=1} \frac{F_{0n}}{n} L^{-n}.$$

Setting $\hat{t}_n = 0$ for $n \geq 1$, it recovers the S function of the one-dimensional dToda hierarchy. Finally, the twistor construction[11] enables us to extract the string equation

$$-1 = \sum_{n=2}^{\infty} n t_n \frac{\partial L}{\partial t_{n-1}} + \sum_{n=1}^{\infty} n \hat{t}_n \frac{\partial L}{\partial \hat{t}_{n-1}}. \quad (4)$$

for the extended dToda hierarchy without referring to the CP^1 matrix model[8].

3 Dispersionless Hirota equations

Proposition 1. *The following relations hold.*

$$F_{n0} = (B_n)_{[0]}, \quad F_{n1} = \text{res}(L^n), \quad n \geq 1 \quad (5)$$

$$F_{\hat{n}0} = (\hat{B}_n)_{[0]}, \quad F_{\hat{n}1} = \text{res}(L^n(\log L - d_n)), \quad n \geq 0 \quad (6)$$

where $(\sum_k a_k p^k)_{[j]} = a_j$.

Proof. From $\log p = \partial S / \partial t_0$ we have

$$\log p = \log L - \sum_{n=1} \frac{F_{0n}}{n} L^{-n} \quad (7)$$

or

$$L = p e^{\sum_{n=1} F_{0n} L^{-n}} = p + F_{01} + \left(-u_1 F_{01} + \frac{1}{2} F_{02} + \frac{1}{2} (F_{01})^2 \right) p^{-1} + O(p^{-2})$$

which yields $u_1 = F_{01}$ and $u_2 = \frac{1}{2} F_{02} - \frac{1}{2} (F_{01})^2$. Therefore, from the p^0 -term of the Lax equation (2) we have $F_{n0} = (B_n)_{[0]}$. On the other hand, from $B_n = \partial S / \partial t_n$ we have

$$B_n = L_+^n = L^n - \sum_{m=1} \frac{F_{nm}}{m} L^{-m}.$$

For $n = 1$, we have $u_2 p^{-1} = \sum_{m=1} F_{1m} L^{-m} / m$ which together with (7) implies $F_{m1} = m u_2 P_{m-1}(F_{0j}/j)$ where $P_m(t)$ are Schur polynomials defined by $e^{\sum_{j=1} t_j z^j} = \sum_{j=0} P_j(t) z^j$. In particular, $u_2 = e^{F_{00}} = F_{11}$. Also, for the p^{-1} -term of the Lax equation (2) we have $F_{n1} = \text{res}(L^n) = u_2 (B_n)_{[1]}$. Furthermore, from $\hat{B}_n = \partial S / \partial \hat{t}_n$ we have

$$\hat{B}_n = [L^n(\log L - d_n)]_+ = L^n(\log L - d_n) - \sum_{m=1} \frac{F_{\hat{n}m}}{m} L^{-m}.$$

The p^{-1} -term gives $F_{\hat{n}1} = \text{res}(L^n(\log L - d_n)) = u_2 (L^n(\log L - d_n))_{[1]}$ where the last equality is due to the identity $\text{res}(L^n(\log L - d_n) dL) = 0$. Finally, from the p^0 -term of the Lax equation (2) we have $F_{\hat{n}0} = (\hat{B}_n)_{[0]}$. \square

Proposition 2. *The two point functions F_{n0} , F_{n1} , $F_{\hat{n}0}$, and $F_{\hat{n}1}$ can be expressed in terms of F_{01} and F_{00} as follows*

$$F_{n0} = \sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{s!s!(n-2s)!} F_{01}^{n-2s} e^{sF_{00}} \quad (8)$$

$$F_{n+1,1} = \sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(n+1)!}{s!(s+1)!(n-2s)!} F_{01}^{n-2s} e^{(s+1)F_{00}} \quad (9)$$

$$F_{\hat{n}0} = \frac{1}{2} \sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{s!s!(n-2s)!} F_{01}^{n-2s} e^{sF_{00}} (F_{00} - 2d_s) \quad (10)$$

$$F_{\widehat{n+1,1}} = \frac{1}{2} \sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(n+1)!}{s!(s+1)!(n-2s)!} F_{01}^{n-2s} e^{(s+1)F_{00}} \left(F_{00} - 2d_s - \frac{1}{s+1} \right). \quad (11)$$

Proof. Using the binomial expansion of powers of L in (5) and the Taylor expansion in (6) with the prescription (3) for $\log L$. \square

We come now to the main result of the work; that is to derive the dHirota equation for the extended dToda hierarchy from the Lax formulation. The result will be expressed in terms of second derivatives of the free energy $F(t_0, t, \hat{t})$.

Theorem 3. *The free energy $F(t_0, t, \hat{t})$ of the extended dToda hierarchy satisfies the following equations*

$$\frac{F_{n+1,m}}{n+1} + \frac{F_{n,m+1}}{m+1} = F_{m,0}F_{n,1} + F_{m,1}F_{n,0}, \quad (n \geq 1, m \geq 1) \quad (12)$$

$$\frac{F_{\widehat{n+1,m}}}{n+1} + \frac{F_{\hat{n},m+1}}{m+1} = F_{m,0}F_{\hat{n},1} + F_{m,1}F_{\hat{n},0}, \quad (m \geq 1, n \geq 0) \quad (13)$$

$$\frac{F_{\widehat{n+1,\hat{m}}}}{n+1} + \frac{F_{\hat{n},\widehat{m+1}}}{m+1} = F_{\hat{m},0}F_{\hat{n},1} + F_{\hat{m},1}F_{\hat{n},0}, \quad (m, n \geq 0). \quad (14)$$

Proof. To prove (12), we note that

$$\begin{aligned} F_{m,n+1,0} &= \frac{\partial F_{n+1,0}}{\partial t_m} = \frac{\partial (L^{n+1})_{[0]}}{\partial t_m} = (n+1) \left(L^n \frac{\partial L}{\partial t_m} \right)_{[0]} \\ &= (n+1) \left((B_n)_{[0]} \frac{\partial u_1}{\partial t_m} + (B_n)_{[1]} \frac{\partial u_2}{\partial t_m} \right) \\ &= (n+1)(F_{n,0}F_{m,1,0} + F_{m,0,0}F_{n,1}) \end{aligned}$$

where $u_1 = F_{01}$ and $u_2 = e^{F_{00}}$ have been used to reach the last equality. Similarly, we have

$$F_{n,m+1,0} = (m+1)(F_{m,0}F_{n,1,0} + F_{n,0,0}F_{m,1}).$$

Hence

$$\frac{F_{n+1,m}}{n+1} + \frac{F_{n,m+1}}{m+1} = F_{m,0}F_{n,1} + F_{m,1}F_{n,0}, \quad (n \geq 1, m \geq 1).$$

Equations (13) and (14) can be verified in a similar manner. \square

Corollary 4. *The two point functions F_{mn} , $F_{\widehat{mn}}$, and $F_{\widehat{\widehat{mn}}}$ are all determined by the fundamental variables F_{00} and F_{01} .*

Proof. This is just an immediate consequence of Proposition 2 and Theorem 3. \square

We shall show later on that the expression of (12)-(14) has a simple interpretation from topological field theory.

4 Catalan numbers and two-point functions

From dispersionless Hirota equations (12)-(14), we see that the building blocks are the two-point functions (5) and (6). Motivated by the work of Kodama and Pierce [12] we like to consider the two-point functions F_{mn} , $F_{\widehat{mn}}$, and $F_{\widehat{\widehat{mn}}}$ in the case with $F_{00} = F_{01} = 0$.

Proposition 5.

$$F_{2k,0} = (k+1)C_k, \quad F_{2k+1,0} = 0 \quad (15)$$

$$F_{2k+1,1} = (2k+1)C_k, \quad F_{2k,1} = 0 \quad (16)$$

$$F_{\widehat{2k},0} = -(k+1)d_k C_k, \quad F_{\widehat{2k+1},0} = 0 \quad (17)$$

$$F_{\widehat{2k+1},1} = -(2k+1) \left(d_k + \frac{1}{2(k+1)} \right) C_k, \quad F_{\widehat{2k},1} = 0 \quad (18)$$

where C_k is the k -th Catalan number defined by

$$C_k = \frac{1}{k+1} \binom{2k}{k}.$$

Proof. This is an immediate consequence by setting $u_1 = F_{01} = 0$ and $u_2 = F_{11} = e^{F_{00}} = 1$ in the equations (8)-(11). \square

Let us derive the two point functions $F_{n,m}$ from the dHirota equation (12).

Theorem 6. (Kodama and Pierce[12]) *The two point function F_{nm} for the extended dToda hierarchy with $F_{00} = 0$ and $F_{01} = 0$ are given by*

$$\begin{aligned} F_{2k,0} &= (k+1)C_k, \quad k = 1, 2, \dots \\ F_{2k+1,2l+1} &= \frac{(2l+1)(2k+1)(l+1)(k+1)}{l+k+1} C_k C_l, \quad k, l = 0, 1, 2, \dots \\ F_{2k,2l} &= \frac{lk(l+1)(k+1)}{l+k} C_k C_l, \quad k, l = 1, 2, \dots \\ F_{nm} &= 0, \quad \text{otherwise} \end{aligned}$$

where C_k is the k -th Catalan number.

Proof. Here we present a derivation of $F_{2k,2l}$ from the dHirota equation (12). Writing $F_{2k,2l}$ in the expression

$$\begin{aligned} F_{2k,2l} &= (F_{2k,2l} + \frac{2l}{2k+1}F_{2k+1,2l-1}) - \frac{2l}{2k+1}(F_{2k+1,2l-1} + \frac{2l-1}{2k+2}F_{2k+2,2l-2}) \\ &\quad + \frac{2l(2l-1)}{(2k+1)(2k+2)}(F_{2k+2,2l-2} + \frac{2l-2}{2k+3}F_{2k+3,2l-3}) + \cdots \\ &\quad + \frac{2l(2l-1)\cdots 3}{(2k+1)(2k+2)\cdots(2k+2l-2)}(F_{2k+2l-2,2} + \frac{2}{2k+2l-1}F_{2k+2l-1,1}) \\ &\quad - \frac{2l!}{(2k+1)(2k+2)\cdots(2k+2l-1)}F_{2k+2l-1,1}. \end{aligned}$$

Then, using the dHirota equation (12), we have

$$\begin{aligned} F_{2k,2l} &= 2l(F_{2k,0}F_{2l-1,1} + F_{2k,1}F_{2l-1,0}) - \frac{2l(2l-1)}{2k+1}(F_{2k+1,0}F_{2l-2,1} + F_{2k+1,1}F_{2l-2,0}) \\ &\quad + \frac{2l(2l-1)(2l-2)}{(2k+1)(2k+2)}(F_{2k+2,0}F_{2l-3,1} + F_{2k+2,1}F_{2l-3,0}) + \cdots \\ &\quad - \frac{2l!}{(2k+1)(2k+2)\cdots(2k+2l-1)}F_{2k+2l-1,1}. \end{aligned}$$

Taking into account (15) and (16) we get

$$\begin{aligned} F_{2k,2l} &= 2l((k+1)C_k(2l-1)C_{l-1}) - \frac{2l(2l-1)}{2k+1}((2k+1)C_k l C_{l-1}) \\ &\quad + \frac{2l(2l-1)(2l-2)}{(2k+1)(2k+2)}((k+2)C_{k+1}(2l-3)C_{l-2}) + \cdots \\ &\quad + \frac{(2l)!}{(2k+1)(2k+2)\cdots(2k+2l-2)}((k+l)C_{k+l-1}) \\ &\quad - \frac{(2l)!}{(2k+1)(2k+2)\cdots(2k+2l-1)}((2k+2l-1)C_{k+l-1}) \\ &= \sum_{i=0}^{l-1} \frac{(2l)!(2k)!}{(2l-2i-2)!(2k+2i)!} [(k+i+1) - (l-i)] C_{k+i} C_{l-i-1} \\ &= \frac{l!k!(l+1)!(k+1)!C_k C_l}{(k+l)!(k+l-1)!} \sum_{i=0}^{l-1} \binom{k+l-1}{i} \left[\binom{k+l}{i+1} - \binom{k+l}{i} \right] \end{aligned}$$

where the formula

$$C_{k+p} = 2^p \frac{(2k+2p-1)!!(k+1)!}{(2k-1)!!(k+p+1)!} C_k$$

for the Catalan numbers has been used. Since for any $p > 1$, we have

$$\begin{aligned} &\sum_{i=0}^{p-1} \binom{k+l-1}{i} \left[\binom{k+l}{i+1} - \binom{k+l}{i} \right] = \sum_{i=1}^p \binom{k+l-1}{i-1} \left[\binom{k+l}{i} - \binom{k+l}{i-1} \right] \\ &= \sum_{i=2}^p \binom{k+l-1}{i-1} \left[\binom{k+l-1}{i} + \binom{k+l-1}{i-1} - \binom{k+l-1}{i-1} - \binom{k+l-1}{i-2} \right] + (k+l-1) \\ &= \binom{k+l-1}{p-1} \binom{k+l-1}{p} \end{aligned}$$

where the Pascal identity $\binom{a}{b} = \binom{a-1}{b} + \binom{a-1}{b-1}$ has been used to reach the second equality.

Hence,

$$\begin{aligned} F_{2k,2l} &= \frac{l!k!(l+1)!(k+1)!C_kC_l}{(k+l)!(k+l-1)!} \binom{k+l-1}{l-1} \binom{k+l-1}{l} \\ &= \frac{lk(l+1)(k+1)}{k+l} C_k C_l. \end{aligned}$$

Substituting $F_{2k,2l}$ into (12) for $n = 2k$ and $m = 2l + 1$, we obtain

$$F_{2k+1,2l+1} = \frac{(2l+1)(2k+1)(l+1)(k+1)}{k+l+1} C_k C_l.$$

This is just the result obtained by Kodama and Pierce in [12]. \square

Corollary 7. *The two point functions F_{mn} for $m, n \geq 0$ are positive-defined, i.e. $F_{mn} \geq 0$.*

Next we deal with the two point function $F_{\widehat{nm}}$.

Theorem 8. *The two point function $F_{\widehat{nm}}$ for the extended dToda hierarchy with $F_{00} = 0$ and $F_{01} = 0$ are given by*

$$\begin{aligned} F_{\widehat{2k},0} &= -(k+1)d_k C_k, \quad k = 1, 2, \dots \\ F_{\widehat{2k+1},2l+1} &= -\frac{(2l+1)(2k+1)(l+1)(k+1)}{l+k+1} \left(d_k + \frac{1}{2(l+k+1)} \right) C_k C_l, \quad k, l = 0, 1, 2, \dots \\ F_{\widehat{2k},2l} &= -\frac{lk(l+1)(k+1)}{l+k} \left(d_k - \frac{l}{2k(l+k)} \right) C_k C_l, \quad k, l = 1, 2, \dots \\ F_{\widehat{nm}} &= 0, \quad \text{otherwise.} \end{aligned}$$

Proof. Following the same procedure by using the dHirota equation (13) and (15)-(18), we have

$$\begin{aligned} F_{\widehat{2k},2l} &= -\sum_{i=0}^{l-1} \frac{(2l)!(2k)!C_{k+i}C_{l-i-1}}{(2l-2i-2)!(2k+2i)!} \left[(k+i+1)d_{k+i} - (l-i) \left(d_{k+i} + \frac{1}{2(k+i+1)} \right) \right] \\ &= (I) + (II) \end{aligned}$$

where

$$\begin{aligned} (I) &= -k!l!(k+1)!(l+1)!C_k C_l \sum_{i=0}^{l-1} \frac{(k-l+2i+1)d_k}{(l-i)!(l-i-1)!(k+i+1)!(k+i)!} \\ (II) &= -k!l!(k+1)!(l+1)!C_k C_l \sum_{i=1}^{l-1} \frac{(k-l+2i+1)(\frac{1}{k+i} + \dots + \frac{1}{k+1}) - \frac{(l-i)}{2(k+i+1)}}{(l-i)!(l-i-1)!(k+i+1)!(k+i)!}. \end{aligned}$$

Part (I) can be computed as before and it gives

$$(I) = -\frac{lk(l+1)(k+1)}{k+l} d_k C_k C_l.$$

While part (II) can be written as follows

$$-\frac{lk(l+1)(k+1)}{k+l}C_kC_l \left[\frac{2(k+l) \sum_{i=1}^{l-1} \binom{k+l-1}{i-1} \left[\binom{k+l}{i} - \binom{k+l}{i-1} \right] \left(\frac{1}{k+l-i} + \cdots + \frac{1}{k+1} \right) - \sum_{i=0}^{l-1} \binom{k+l}{i}^2}{2(k+l) \binom{k+l-1}{l} \binom{k+l-1}{k}} \right]$$

where the first summation of the numerator in the bracket can be simplified as

$$\begin{aligned} & \sum_{i=1}^{l-1} \binom{k+l-1}{i-1} \left[\binom{k+l}{i} - \binom{k+l}{i-1} \right] \left(\frac{1}{k+l-i} + \cdots + \frac{1}{k+1} \right) \\ &= \frac{1}{k+1} \sum_{i=1}^{l-1} \binom{k+l-1}{i-1} \left[\binom{k+l}{i} - \binom{k+l}{i-1} \right] + \frac{1}{k+2} \sum_{i=1}^{l-2} \binom{k+l-1}{i-1} \left[\binom{k+l}{i} - \binom{k+l}{i-1} \right] \\ &+ \cdots + \frac{1}{k+l-1} \sum_{i=1}^1 \binom{k+l-1}{i-1} \left[\binom{k+l}{i} - \binom{k+l}{i-1} \right] \\ &= \frac{1}{k+1} \binom{k+l-1}{l-1} \binom{k+l-1}{l-2} + \frac{1}{k+2} \binom{k+l-1}{l-2} \binom{k+l-1}{l-3} \\ &+ \cdots + \frac{1}{k+l-1} \binom{k+l-1}{1} \binom{k+l-1}{0} \\ &= \sum_{i=1}^{l-1} \frac{1}{k+l-i} \binom{k+l-1}{i-1} \binom{k+l-1}{i}. \end{aligned}$$

Hence the numerator in the bracket is given by

$$\begin{aligned} & 2(k+l) \sum_{i=1}^{l-1} \frac{1}{k+l-i} \binom{k+l-1}{i-1} \binom{k+l-1}{i} - \sum_{i=0}^{l-1} \binom{k+l}{i}^2 \\ &= \sum_{i=1}^{l-1} \left[2 \binom{k+l-1}{i-1} \binom{k+l}{i} - \binom{k+l}{i}^2 \right] - \binom{k+l}{0}^2 \\ &= \sum_{i=1}^{l-1} \binom{k+l}{i} \left[2 \binom{k+l-1}{i-1} - \binom{k+l-1}{i} - \binom{k+l-1}{i-1} \right] + 1 \\ &= \sum_{i=1}^{l-1} \left[\binom{k+l-1}{i} + \binom{k+l-1}{i-1} \right] \left[\binom{k+l-1}{i-1} - \binom{k+l-1}{i} \right] + 1 \\ &= - \binom{k+l-1}{l-1}^2 \end{aligned}$$

which implies

$$F_{\widehat{2k,2l}} = -\frac{lk(l+1)(k+1)}{l+k} \left(d_k - \frac{l}{2k(l+k)} \right) C_k C_j.$$

Substituting $F_{\widehat{2k,2l}}$ into (13) for $n = 2k$ and $m = 2l + 1$, we obtain

$$F_{\widehat{2k+1,2l+1}} = -\frac{(2l+1)(2k+1)(l+1)(k+1)}{k+l+1} \left(d_k + \frac{1}{2(l+k+1)} \right) C_k C_l.$$

□

Corollary 9. *The two point functions $F_{\widehat{mn}}$ for $mn \neq 0$ are negative-defined, i.e. $F_{\widehat{mn}} < 0$.*

Proof. The only case to be considered is $F_{\widehat{2k}, 2l}$ in which

$$d_k - \frac{l}{2k(l+k)} = \left(d_k - \frac{1}{2k}\right) + \frac{1}{2(k+l)} > 0.$$

□

Finally, we compute the two point function $F_{\widehat{n}\widehat{m}}$.

Theorem 10. *The two point function $F_{\widehat{n}\widehat{m}}$ for the extended dToda hierarchy with $F_{00} = 0$ and $F_{01} = 0$ are given by*

$$\begin{aligned} F_{\widehat{2k}, \widehat{0}} &= -\frac{k+1}{2} d_k C_k, \quad k = 1, 2, \dots \\ F_{\widehat{2k+1}, \widehat{2l+1}} &= \frac{(2l+1)(2k+1)(l+1)(k+1)}{l+k+1} \times \\ &\quad \left[\left(d_k + \frac{1}{2(l+k+1)}\right) \left(d_l + \frac{1}{2(l+k+1)}\right) + \frac{1}{4(k+l+1)^2} \right] C_k C_l, \quad k, l = 0, 1, 2, \dots \\ F_{\widehat{2k}, \widehat{2l}} &= \frac{lk(l+1)(k+1)}{l+k} \left[\left(d_k - \frac{l}{2k(l+k)}\right) \left(d_l - \frac{k}{2l(l+k)}\right) + \frac{1}{4(k+l)^2} \right] C_k C_l, \quad k, l = 1, 2, \dots \\ F_{\widehat{n}\widehat{m}} &= 0, \quad \text{otherwise.} \end{aligned}$$

Proof. Using the dHirota equation (14) and taking into account (17)-(18) we have

$$\begin{aligned} F_{\widehat{2k}, \widehat{2l}} &= \sum_{i=0}^{l-1} \frac{(2l)!(2k)!C_{k+i}C_{l-i-1}}{(2l-2i-2)!(2k+2i)!} \times \\ &\quad \left[(k+i+1)d_{k+i}(d_{l-i-1} + \frac{1}{2(l-i)}) - (d_{k+i} + \frac{1}{2(k+i+1)})(l-i)d_{l-i-1} \right] \\ &\quad - \frac{2l!2k!}{2(2k+2l)!} (k+l+1)d_{k+l}C_{k+l} \\ &= \frac{l!k!(l+1)!(k+1)!C_k C_l}{2((k+l)!)^2} [(I) + (II) + (III) + (IV) + (V)] \end{aligned}$$

where

$$\begin{aligned} (I) &= 2((k+l)!)^2 \sum_{i=0}^{l-1} \frac{\left[(k-l+2i+1)(d_k + (\frac{1}{k+i} + \dots + \frac{1}{k+1})) - \frac{(l-i)}{2(k+i+1)} \right] d_l}{(l-i)!(l-i-1)!(k+i+1)!(k+i)!} \\ (II) &= 2((k+l)!)^2 \sum_{i=0}^{l-1} \frac{\left[-(k-l+2i+1)(\frac{1}{l-i} + \dots + \frac{1}{l}) + \frac{(k+i+1)}{2(l-i)} \right] d_k}{(l-i)!(l-i-1)!(k+i+1)!(k+i)!} \\ (III) &= 2((k+l)!)^2 \sum_{i=0}^{l-1} \frac{\left[-(k-l+2i+1)(\frac{1}{k+i} + \dots + \frac{1}{k+1})(\frac{1}{l-i} + \dots + \frac{1}{l}) \right]}{(l-i)!(l-i-1)!(k+i+1)!(k+i)!} \\ (IV) &= 2((k+l)!)^2 \sum_{i=0}^{l-1} \frac{\left[\frac{(k+i+1)}{2(l-i)}(\frac{1}{k+i} + \dots + \frac{1}{k+1}) + \frac{(l-i)}{2(k+i+1)}(\frac{1}{l-i} + \dots + \frac{1}{l}) \right]}{(l-i)!(l-i-1)!(k+i+1)!(k+i)!} \end{aligned}$$

$$(V) = -d_{k+l}.$$

Each term can be calculated as follows:

$$\begin{aligned}
(I) &= 2(k+l) \binom{k+l-1}{k} \binom{k+l-1}{l} \left[d_k - \frac{l}{2k(l+k)} \right] d_l \\
(II) &= d_k - 2(k+l) \binom{k+l-1}{k} \binom{k+l-1}{l} \left[\frac{d_k k}{2l(k+l)} \right] \\
(III) &= - \sum_{j=0}^{l-1} \frac{1}{j+1} \sum_{i=0}^j \binom{k+l}{i}^2 - \sum_{j=1}^{l-1} \frac{1}{k+l-j} \sum_{i=1}^j \binom{k+l}{i}^2 \\
&\quad + \frac{1}{k+l} \binom{k+l-1}{l-1} \binom{k+l-1}{l} + \sum_{i=0}^{l-1} \frac{1}{k+l-i} \\
(IV) &= \sum_{j=1}^{l-1} \frac{1}{k+l-j} \sum_{i=1}^j \binom{k+l}{i}^2 + \sum_{j=0}^{l-1} \frac{1}{j+1} \sum_{i=0}^j \binom{k+l}{i}^2 \\
(V) &= -d_k - \sum_{i=0}^{l-1} \frac{1}{k+l-i}.
\end{aligned}$$

It turns out that

$$F_{\widehat{2k}, \widehat{2l}} = \frac{lk(l+1)(k+1)}{l+k} \left[\left(d_k - \frac{l}{2k(l+k)} \right) \left(d_l - \frac{k}{2l(l+k)} \right) + \frac{1}{4(k+l)^2} \right] C_k C_l.$$

Substituting $F_{\widehat{2k}, \widehat{2l}}$ into (14) for $n = 2k$ and $m = 2l + 1$, we obtain

$$\begin{aligned}
F_{\widehat{2k+1}, \widehat{2l+1}} &= \frac{(2l+1)(2k+1)(l+1)(k+1)}{l+k+1} \times \\
&\quad \left[\left(d_k + \frac{1}{2(l+k+1)} \right) \left(d_l + \frac{1}{2(l+k+1)} \right) + \frac{1}{4(k+l+1)^2} \right] C_k C_l.
\end{aligned}$$

□

Corollary 11. *The two point functions $F_{\widehat{m}\widehat{n}}$ for $mn \neq 0$ are positive-defined, i.e. $F_{\widehat{m}\widehat{n}} \geq 0$.*

5 Back to the topological CP^1 model

The relationship between integrable systems and topological field theories has dramatic advances in the past two decades (see e.g. [1, 3, 4, 13, 14, 21, 22]). For the extended dToda hierarchy the corresponding topological field is described by two primary fields (or observables) $\{\mathcal{O}_1 = 1 \in H^0(CP^1), \mathcal{O}_2 = \omega \in H^2(CP^1)\}$ with coupling parameters $T^{\alpha,0}, \alpha = 1, 2$. When the theory couples to topological gravity, a set of new variables emerge as gravitational descendants $\{\sigma_n(\mathcal{O}_\alpha)\}$ with new coupling constants $\{T^{\alpha,n}\}$. The identity operator now becomes the puncture operator $\mathcal{O}_1 = P$ and we also denote $\mathcal{O}_2 = Q$. The space spanned by $\{T^{\alpha,n}, n = 0, 1, 2, \dots\}$ is called the full phase

space and the subspace parametrized by $T^{\alpha,0}$ the small phase space. The generating function of correlation function is the full free energy defined by

$$\mathcal{F}(T) = \sum_{g=0} \mathcal{F}_g = \sum_{g=0}^{\infty} \langle e^{\sum_{\alpha,n} T^{\alpha,n} \sigma_n(\mathcal{O}_{\alpha})} \rangle_g.$$

Since the free energy $F(t_0, t, \hat{t})$ of the extended dToda hierarchy corresponds to the genu-zero generating function \mathcal{F}_0 of CP^1 under the identification

$$t_{n+1} = \frac{T^{2,n}}{(n+1)!}, \quad \hat{t}_n = \frac{2T^{1,n}}{n!}, \quad n \geq 0.$$

where $\hat{t}_0 = 2t_0 = 2T^{1,0} = 2x$. Hence a generic genus-zero m -point correlation function can be calculated as follows

$$\langle \sigma_{n_1}(\mathcal{O}_{\alpha_1}) \sigma_{n_2}(\mathcal{O}_{\alpha_2}) \cdots \sigma_{n_m}(\mathcal{O}_{\alpha_m}) \rangle = \frac{\partial^m F}{\partial T^{\alpha_1, n_1} \partial T^{\alpha_2, n_2} \cdots \partial T^{\alpha_m, n_m}}.$$

In particular, the metric on the space of primary fields is defined by three-point correlation function $\eta_{\alpha\beta} = \langle P \mathcal{O}_{\alpha} \mathcal{O}_{\beta} \rangle$ with $\eta_{11} = \eta_{22} = 0$ and $\eta_{12} = \eta_{21} = 1$, and hence $\mathcal{O}_1 = \mathcal{O}^2$ and $\mathcal{O}_2 = \mathcal{O}^1$.

The Lax equations of the extended dToda hierarchy can be written as

$$\frac{\partial L}{\partial T^{\alpha,n}} = \{B_{\alpha,n}, L\}, \quad \alpha = 1, 2; n = 0, 1, 2, \dots$$

where

$$B_{1,n} = \frac{2}{n!} (L^n (\log L - d_n))_{\geq 0}, \quad B_{2,n} = \frac{1}{(n+1)!} (L^{n+1})_{\geq 0}$$

and the string equation (4) becomes

$$0 = 1 + \sum_{n=1}^{\infty} T^{2,n} \frac{\partial L}{\partial T^{2,n-1}} + \sum_{n=1}^{\infty} T^{1,n} \frac{\partial L}{\partial T^{1,n-1}}.$$

Shifting the variable $T^{1,1} \rightarrow T^{1,1} - 1$ we have

$$\frac{\partial L}{\partial T^{1,0}} = 1 + \sum_{n=1}^{\infty} T^{2,n} \frac{\partial L}{\partial T^{2,n-1}} + \sum_{n=1}^{\infty} T^{1,n} \frac{\partial L}{\partial T^{1,n-1}}$$

which, after extracting the p^0 term, yields

$$\begin{aligned} t^1(T) &= T^{1,0} + \sum_{\alpha} \sum_{n=1}^{\infty} T^{\alpha,n} \langle \sigma_{n-1}(\mathcal{O}_{\alpha}) Q \rangle, \\ t^2(T) &= T^{2,0} + \sum_{\alpha} \sum_{n=1}^{\infty} T^{\alpha,n} \langle \sigma_{n-1}(\mathcal{O}_{\alpha}) P \rangle \end{aligned}$$

where we identify the flat coordinate $t^{\alpha} = \langle P \mathcal{O}^{\alpha} \rangle$ as

$$t^1 = u_1 = \langle PQ \rangle, \quad t^2 = \log u_2 = \langle PP \rangle.$$

Therefore, in small space $t^1 = T^{1,0}$ and $t^2 = T^{2,0}$. The condition $F_{01} = F_{00} = 0$ then corresponds to $T^{\alpha,n} = 0$, $\forall \alpha, n$. In the Landau-Ginzburg formulation of the topological CP^1 model, it can be shown [21] that the following genus-zero topological recursion relation holds.

$$\langle \sigma_n(\mathcal{O}_\alpha)XY \rangle = \sum_{\beta} \langle \sigma_{n-1}(\mathcal{O}_\alpha)\mathcal{O}^\beta \rangle \langle \mathcal{O}_\beta XY \rangle. \quad (19)$$

Proposition 12. *The genus-zero topological recursion relation (19) implies the dHirota equations (12)-(14).*

Proof. Using (19) we have

$$\begin{aligned} & \frac{\partial}{\partial T^{1,0}} [\langle \sigma_{n+1}(Q)\sigma_m(Q) \rangle + \langle \sigma_n(Q)\sigma_{m+1}(Q) \rangle] \\ &= \langle \sigma_{n+1}(Q)\sigma_m(Q)P \rangle + \langle \sigma_n(Q)\sigma_{m+1}(Q)P \rangle \\ &= \frac{\partial}{\partial T^{1,0}} [\langle \sigma_n(Q)Q \rangle \langle \sigma_m(Q)P \rangle + \langle \sigma_n(Q)P \rangle \langle \sigma_m(Q)Q \rangle] \end{aligned}$$

which, after integrating over $T^{1,0}$, implies

$$\langle \sigma_{n+1}(Q)\sigma_m(Q) \rangle + \langle \sigma_n(Q)\sigma_{m+1}(Q) \rangle = \langle \sigma_n(Q)Q \rangle \langle \sigma_m(Q)P \rangle + \langle \sigma_n(Q)P \rangle \langle \sigma_m(Q)Q \rangle.$$

Similarly, we have

$$\begin{aligned} \langle \sigma_{n+1}(P)\sigma_m(Q) \rangle + \langle \sigma_n(P)\sigma_{m+1}(Q) \rangle &= \langle \sigma_n(P)Q \rangle \langle \sigma_m(Q)P \rangle + \langle \sigma_n(P)P \rangle \langle \sigma_m(Q)Q \rangle, \\ \langle \sigma_{n+1}(P)\sigma_m(P) \rangle + \langle \sigma_n(P)\sigma_{m+1}(P) \rangle &= \langle \sigma_n(P)Q \rangle \langle \sigma_m(P)P \rangle + \langle \sigma_n(P)P \rangle \langle \sigma_m(P)Q \rangle. \end{aligned}$$

The proof is completed by noting the following identifications:

$$\begin{aligned} \langle \sigma_m(P)\sigma_n(P) \rangle &= \frac{4F_{\hat{m}\hat{n}}}{m!n!}, \quad m, n \geq 0 \\ \langle \sigma_m(P)\sigma_{n-1}(Q) \rangle &= \frac{2F_{\hat{m}n}}{m!n!}, \quad m \geq 0, n \geq 1 \\ \langle \sigma_{m-1}(Q)\sigma_{n-1}(Q) \rangle &= \frac{F_{mn}}{m!n!}, \quad m, n \geq 1. \end{aligned}$$

□

We thus show that the integrable structure associated with the genus-zero topological CP^1 model is the extended dToda hierarchy. Furthermore, integrating the two point functions $\langle \sigma_n(P)P \rangle$ and $\langle \sigma_n(Q)P \rangle$ over $T^{1,0}$ we obtain the one-point functions

$$\langle \sigma_n(P) \rangle = \frac{2}{(n+1)!} F_{\widehat{n+1},0}, \quad \langle \sigma_n(Q) \rangle = \frac{1}{(n+2)!} F_{n+2,0}.$$

In particular, their values in the limit of zero couplings ($T^{\alpha,n} = 0 \forall \alpha, n$) are

$$\langle \sigma_{2k-1}(P) \rangle = -\frac{2d_k}{(k!)^2}, \quad \langle \sigma_{2k-2}(Q) \rangle = \frac{1}{(k!)^2}.$$

6 Concluding remarks

We have introduced the extended dToda hierarchy from the one-dimensional dToda hierarchy by adding logarithmic flows. The full hierarchy equations of the extended dToda system can be summarized by a set of dHirota equations which involve second derivatives of the free energy F in time parameters t_0 , t_n and \hat{t}_n . Based on these dHirota equations we computed the two point functions $F_{n,m}$, $F_{\hat{n},m}$, and $F_{\hat{n},\hat{m}}$ in the case with $F_{00} = F_{01} = 0$. Our results extend the previous formula obtained by Kodama and Pierce for the one-dimensional dToda system to those results for the extended dToda system. Furthermore, we have shown that, in terms of CP^1 time parameters, the dHirota equations are nothing but a direct consequence of the genus-zero topological recursion relations. This provides another route to realize that the integrable structure associated with the topological CP^1 model at genus-zero level is the extended dToda hierarchy.

There are two remarks in order. First, Milanov [17] has studied the Hirota quadratic equations associated with the extended Toda hierarchy by constructing some vertex operators taking values in the algebra of differential operators on the affine line. The peculiar properties of these Hirota equations have been studied in some recent works [16, 19]. It would be interesting to investigate the dispersionless limit of the Hirota quadratic equations. Second, in [12] a combinatorial meaning of the two point functions F_{nm} has been investigated from large- N expansion of unitary ensemble of random matrices. It is quite natural to ask how to realize the geometric/topological meaning of the rational numbers $F_{\hat{n},m}$ and $F_{\hat{n},\hat{m}}$ from the CP^1 matrix integral[8] which contains extra logarithmic terms. We hope to back to all these issues in our future works.

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References

- [1] Aoyama, S., Kodama, Y.: Topological Landau-Ginzburg theory with a rational potential and the dispersionless KP hierarchy. *Commun. Math. Phys.* **182**, 185–219 (1996)
- [2] Carlet, G., Dubrovin, B., Zhang, Y.: The extended Toda hierarchy, *Mosc. Math. J.* **4**, 313-332 (2004)
- [3] Dijkgraaf, R.: Intersection theory, integrable hierarchies and topological field theory, In: J. Fröhlich *et al* (eds), *New Symmetry Principles in Quantum Field Theory*, New york: Plenum, 1993, 95–158
- [4] Dubrovin, B.: Geometry of 2D topological field theories, In: M. Francaviglia and S. Greco, (eds), *Integrable Systems and Quantum Group*, Springer, Berlin, 1996, pp. 120–348

- [5] Dubrovin, B., Zhang, Y.J.: Normal forms of integrable PDEs, Frobenius manifolds and Gromov - Witten invariants, Preprint, math/0108160.
- [6] Dubrovin, B., Zhang, Y.J.: Virasoro symmetries of the extended Toda hierarchy, *Comm. Math. Phys.* **250**, 161V193 (2004)
- [7] Eguchi, T., Hori, K., Yang, S.K., Topological σ -models and large- N matrix integral, *Internat. J. Modern Phys. A* **10**, 4203V4224 (1995)
- [8] Eguchi, T., Yang, S. K.; The topological CP^1 model and the large- N matrix integral, *Mod. Phys. Lett. A* **9**, 2893–2902 (1994)
- [9] Hori, K., Katz, S., Klemm, A., Pandharipande, R., Thomas, R., Vafa, C., Vakil, R., Zaslow, E.: *Mirror symmetry*, Clay Mathematics Monographs, 1, American Mathematical Society, (2003).
- [10] Jackson, D. M.; On an integral representation for the genus series for 2-cell embeddings, *Trans. Amer. Math. Soc.* **344**, 755V772 (1994)
- [11] Kanno, H. and Ohta, Y.; Topological strings with scaling violation and Toda lattice hierarchy, *Nucl. Phys. B* **442**, 179-201 (1995).
- [12] Kodama, Y., Pierce, V.; Combinatorics of dispersionless integrable systems and universality in random matrix theory, *Commun. Math. Phys.* **292**, 529–568 (2009).
- [13] Kontsevich, M.; Intersection theory on the moduli space of curves and the matrix Airy function, *Comm. Math. Phys.* **147**, 1-23 (1992)
- [14] Krichever, I.: The dispersionless lax equations and topological minimal models, *Commun. Math. Phys.* **143**, 415–429 (1992).
- [15] Kwak, J. H., Lee, J.: Genus polynomials of dipoles, *Kyungpook Math. J.* **33**, 115V125 (1993)
- [16] Li, C.Z., He, J.S., Wu, K., Cheng, Y.: Tau function and Hirota bilinear equations for the extended bigraded Toda hierarchy, *J. Math. Phys.* **51** 043514 (2010)
- [17] Milanov, T.E.: Hirota quadratic equations for the extended Toda hierarchy, *Duke Math. J.* **138**, 161-178 (2007)
- [18] Stanley, R.P.: *Enumerative Combinatorics, Volume 1*, 2nd edition, Cambridge University Press (2000)
- [19] Takasaki, K.: Two extensions of 1D Toda hierarchy, *J. Phys. A* **43** 434032 (2010)
- [20] Takasaki, K., Takebe, T.: Integrable hierarchies and dispersionless limit, *Rev. Math. Phys.* **7**, 743–808 (1995).

- [21] Witten, E.: On the structure of the topological phase of two-dimensional gravity, Nucl. Phys. B **340**, 281–332 (1990)
- [22] Witten, E.: Two dimensional gravity and intersection theory on moduli space, Surveys in Diff. Geom. **1** (1991) 243–310
- [23] Zhang, Y.: On the CP1 topological sigma model and the Toda lattice hierarchy, J. Geom. Phys. **40** 215V232 (2002)